COURSE 2- RINGS

LECTURE 1

Example: (0) is a Maximal ideal of R.

Example: $((0)\,,+_{12}\,,._{12})$ is a maximal ideal of the ring $(Z_{17}\,,+_{17}\,,._{17})$.

Q: find all Maximal ideal of Z_{12}

Solution: above its (**Pe.4**)

Remark: In general Z, we have (p) Maximal ideal in Z where p is a prime number.

Theorem: In the ring $(Z,+,.)$, $((p),+,.)$ is a maximal ideal ⟺ p is a prime number.

Proof:

⇒ suppose that (P) is a maximal

ideal of $(Z,+,.)$ T.P: P is a

prime number??

Let P is not prime number

$\therefore P = k_1.k_2\,;\,1 < k_1$

$< P\ \ ,1 < k_2 < P\ \because$

$k_1 < P \Longrightarrow (P) \subset$

$(k_1) \rightarrow C!$

$\because k_2 < P \Longrightarrow (P) \subset (k_1) \rightarrow C!$

∴ P is a prime number

$\Leftarrow$ We have P

is a prime

number. T.P:

(P) is a

Maximal ideal?

Since P is a prime number, then all the ideal are (2) , (3) ,(5),(7) , (11) , ...

$$0 \subset (4)$$

$$\subset (2) \rightarrow$$

$$Maxima$$

$$l\ 0 \subset$$

$$(6) \subset$$

$$(3) \rightarrow$$

$$Maxima$$

$$l$$

$$0 \subset (10) \subset (5) \rightarrow Maximal$$

Theorem: Let R be a comm. Ring with 1. Every proper ideal contains in maximal ideal.

Proof:

Let I ideal of $R \ni I \neq R$

$A = \{J : I \subseteq J \text{ and } J \text{ ideale of } R\}$

$A \neq \emptyset$ Choose $I \subset J_1 \subset J_2 \subset \cdots$

$\therefore$ By (Zorn’s Lemma); J is maximal

LECTURE 2

Theorem: Let $(I, +, .)$ be only Maximal ideal of R. then there only 1 , 0 are idempotent elements

Proof:

Let $x \in R \ni x \neq 1, \neq 0$

If $x^2 = x \Longrightarrow x^2 = x$

$x^2 - x = 0$

$x(x - 1) = 0$

$x = 0 \;\; or \;\; x = 1 \rightarrow C!$

OR

Let x any ideal

And x-1 another ideal

$x \notin$ Maximal

$(x) \subset I$ and $(x - 1) \subset I$

From definition first conditional of ideal

$x - x + 1 = 1 \in I \rightarrow I = R \rightarrow C!$

Theorem: If $(I, +, .)$ Is an ideal of comm. Ring with 1 $(R, +, .)$, then I is a Maximal ideal $\Longleftrightarrow$ $(\frac{R}{I}, +, .)$ Is a field.

Proof:

$\Rightarrow$ Let I be a

Maximal ideal

of R T.P: $(\frac{R}{I},$

$+ , .)$ Is a

field??

T.P: $\frac{R}{I}$ comm. rang with 1 and every element non zero has inverse

Since: R is comm. $\Longrightarrow \frac{R}{I}$ is comm.

R has 1 $\Longrightarrow \frac{R}{I}$ has $1 + I$

Let $I = 0 + I \neq a + I \in \frac{R}{I}$

$a + I \neq \emptyset \Rightarrow a \notin I$

$\because$ I Maximal $\Rightarrow (a) + I \in R$

$1 \in R \Longrightarrow 1 \in (a) + I$

$\Longrightarrow 1 = r(a) + i \,;\, i \in I \,,\, r \in R$

$\Longrightarrow 1 = 1 - ra$

$\because i \in I \Longrightarrow 1 - ra \in I \Longrightarrow 1 + I \in ra + I =$

$(r + I).(a + I) \Longrightarrow (a + I)^{-1} = r + I \Longrightarrow \frac{R}{I}$

field.

$\Longleftarrow$ Let $(\frac{R}{I}, + , .)$ Be a field

T.P: I is a Maximal ideal of R??

Suppose that J is an ideal

of R $\ni I \subset J \subseteq R$ T.P: $J =$

R??

$\because I \subset J \Longrightarrow \exists\; a \in J \,,\, a \notin I$

LECTURE 3

$\Longrightarrow a + I \neq I \Longrightarrow a + I \neq 0 + I$

$\because \frac{R}{I}$ field

$\exists\ b + I \in \frac{R}{I} \ni (a + I)(b + I) = 1 + I$

$\Longrightarrow 1 - a.b \in J \subset I \subset J$

$\because a \in J\ , b \in R \Longrightarrow a.b \in J$

$\Longrightarrow 1 \in J \Longrightarrow J = R$ why??

$\therefore$ I is a maximal of R

Def: Let R be a ring (commutative with 1). An ideal I is called prime if:-

$$a.b \in I \rightarrow either\ a \in I\ or\ b \in I\ ; \forall a, b \in R$$

Ex: Let $(Z, +,$

$.\,)$ be a ring. Then (P) is prime ideal such that P is a prime number.

$(5) = \{\dots, -10, -5, 0, 5, 10, \dots\}$

Is a prime ideal

$(3) = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$

Is a prime ideal

Remark: $(Z, +, .\,)$ is a prime ideal

$(\{0\}, +, .\,)$ is a prime ideal **In general**

Ex: $(Z_{14}, +_{14}, ._{14})$ is a ring but $(\{0\}, +_{14}, ._{14})$ is not prime ideal because

$2, 7 \in Z_{14}$

$2 \cdot_{14} 7 = 0$ .But $2 \notin$

$\{0\}$ and $7 \notin \{0\}$

Now:

First find all ideals

$2(7) = 14$

2	14
7	7
	1

$I_2 = (2) = \{0 , 2 , 4 , 6 , 8 ,10 , 12 \}$

$I_2 = (7) = \{0 , 7 \}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	0	2	4	6	8	10	12	0	2	4	6	8	10	12
3	0	3	6	9	12	1	4	7	10	13	2	5	8	11
4	0	4	8	12	2	6	10	0	4	8	12	2	6	10
5	0	5	10	1	6	11	2	7	12	3	8	13	4	9
6	0	6	12	4	10	2	8	0	6	12	4	10	2	8
7	0	7	0	7	0	7	0	7	0	7	0	7	0	7
8	0	8	2	10	4	12	6	0	8	2	10	4	12	6
9	0	9	4	13	8	3	12	7	2	11	6	1	10	5
10	0	10	6	2	12	8	4	0	10	6	2	12	8	4
11	0	11	8	5	2	13	10	7	4	1	12	9	6	3
12	0	12	10	8	6	4	2	0	12	10	8	6	4	2
13	0	13	12	11	10	9	8	7	6	5	4	3	2	1

$1 \cdot_{14} 3 = 3 \notin (2)$

Then there is not prime ideal in Z_{14}

Q: Let R be a ring (comm. with1). Then R is an integral domain ⇔ {0} is prime Solution:

⇒ Let R be an integral domain.

$\therefore a.b = 0 \rightarrow a = 0 \ or \ b = 0 \ \therefore \{0\}$ prime.

Why?

⇐ Let {0} is prime

$\therefore a.b = 0 \rightarrow a.b \in \{0\} \rightarrow a \in \{0\} \ or \ b \in \{0\} \quad \forall a, b \in R$

$a=0$ $\quad$ $b=0$

∴ R integral domain.

LECTURE 4

Theorem: Let $(I, +, .)$ Be a prime ideal of a comm. ring R with 1. Then the quotient ring (R/I) is integral domain if and only if $(I, +, .)$ Is a prime ideal.

Proof: ⇒

T.P: I is a prime ideal ??

Let $a, b \in R \ni a.b \in I$.

We must prove that $a \in I \ or \ b \in I$??

$\because a.b \in I \rightarrow a.b + I = I \ (a + I)(b + I) = I = 0 + I$.

But $\frac{R}{I}$ integral domain $\rightarrow \frac{R}{I}$ has no zero divisors

$\rightarrow a + I = I \;\; or \;\; b + I = I$. Why ??

$\rightarrow a \in I \quad or \quad b \in I$

$\rightarrow I$??

$\Leftarrow$

Since R comm. $\rightarrow \frac{R}{I}$ comm. Since R with 1 $\rightarrow \frac{R}{I}$ with $1 + I$. We must prove $\frac{R}{I}$ has no zero divisors.

Let $a + I$, $b + I \in \frac{R}{I} \ni a, b \in R$

$(a + I)(b + I) = 0 + I = I$

$(a.b) + I = I \rightarrow a.b \in I$

But I prime ideal $\rightarrow$ either $a \in I$ or $b \in I$

If: $a \in I \rightarrow a + I = I \rightarrow a + I = 0 + I$

Or: $b \in I \rightarrow b + I = I \rightarrow b + I = 0 + I$

$\rightarrow \frac{R}{I}$ has no zero divisor $\rightarrow \frac{R}{I}$ integral domain

Corollary: Every maximal ideal is prime ideal.

Proof:

Suppose that A is a maximal ideal of R

T.P: A is a prime ideal ??

Suppose $x.y \in A \;\; \forall x, y \in A$? ?

If $x \notin A$:-

Since A maximal , $x \notin A \rightarrow A + (x) = R$

But $1 \in R \rightarrow 1 \in A + (x)$

$\rightarrow 1 = a + rx$, $r \in R$, $a \in A$

$\therefore y = ay + y(rx)$

$= ay + rxy$

But $y \in R; a \in A \to ay \in A$

But $r \in R; x.y \in A \to r(xy) \in A$

$\to ay + r(xy) \in A \to y \in A \to A$ prime ideal

LECTURE 5

Def: We say R principal ideal ring (P.I.R) if every if every ideal of R is principal (**it is generated by one element**)

Ex: (2) is principle in Z

Ex: In Z every ideal is principal. $(a) = \{r.a : r \in R\}$:1

Remark: Z is a P.I.R

Def: if R **integral domain** and every ideal of R is **principal** then R is called principal integral domain (**P.I.D**)

Theorem: Let R be a P.I.D. every non trivial ideal A is prime ⟺ A is maximal

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Remarks:

1. If R is P.I.D, then every non trivial ideal I is prime ⟺ is maximal
2. In $(Z, +, .)$ Every non trivial ideal I is maximal ⟺ I prime

3. If R is P.I.D, then the ideal $((a), +, .)$ is prime (maximal) in R $\Leftrightarrow$ a prime number 4. $(Z, +, .)$ Is P.I.D (By (3))

Def: Let R be a ring. Then the radical of R (Rad(R)) defined by:

$$Rad(R) = \cap \{M: M \text{ is Maximal ideal of } R\}$$

- $Rad(R) \neq \emptyset$
- $Rad(R) \subseteq R$
-

Ans: $Rad(Z) = \cap\{all\ Maximal\ ideals\} \ni (P)$ prime

$\therefore (2) \cap (3) \cap (5) \cap (7) \cap (11) \cap (13) \cap$ …

Q: find Rad (Z_{12})

Solution:

We must find all maximal ideals

Where $12 = (2)(6) = (3)(4) = (1)(12)$

$I_1 = (2) = \{0,2,4,6,8,10\}$

$I_2 = (3) = \{0,3,6,9\}$

$I_3 = (4) = \{0,4,8\}$

$I_4 = (6) = \{0,6\}$

I_4 and I_3 are not maximal ideals because $\subset I_1$. But I_1 and I_2 are both maximal ideal because there not ideal contains them

Then $Rad(Z_{12}) = I_1 \cap I_2 = \{0,2,4,6,8,10\} \cap \{0,3,6,9\} = \{0,6\}$

Def: We say the ring R is semi simple if $Rad(R) = 0$

Def: Let I be an ideal of R. Then $\sqrt{I} = \{r \in R\ ; r^n \in I\ , n \in Z^+\}$ **Remark:**

1. $\sqrt{I} \subseteq R$
2. $\sqrt{I} \neq \emptyset$ why?

LECTURE 7

Thereon: Let I be an ideal of R. Then if J an ideal of $R \Longrightarrow \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

Proof:

Suppose that $x \in \sqrt{I \cap J}$

$\therefore x^n \in \cap I\ J\ ; n \in Z^+$

$\therefore \in \wedge\ ^n \in\ x^n\ I\ x\ J$

$y^n \in I\ , y^m J\ ,$ $n, m \in Z^+$

$\therefore x \in \sqrt{I} \cap x \in \sqrt{J}$

$y^n . y^m \in I\ ,$ because $y^n \in I\ , y^m \in R$

$\therefore \sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$

$\therefore y^{n+m} \in I$

Suppose $y \in \sqrt{I} \cap \sqrt{J}$

$\therefore y^k \in I$

$y \in \sqrt{y}\ , y \in \sqrt{J}$

Also, $y^n . y^m \in J$ because $y^m \in J\ , y^n \in R$

$\therefore y^{n+m} \in J \rightarrow y^k \in J$

$\therefore y^k \in \wedge \in I\ y^k\ J$

$\therefore y^k \in \cap I\ J$

$\therefore y \in \sqrt{I \cap J}$

$\therefore \sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I \cap J}$

$\therefore \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$

.

Def: Let $f: R_1 \to R_2$ be a function we say f is ring homomorphism if

1. $f(a+b) = f(a) +_2 f(b)$
2. $f(a.b) = f(a) \cdot_2 f(b)$

Example: $f: R_1 \to R_2 \ni f(a) = \overline{0}, a \in R$

Solution:

$$f(a+b) = \overline{0} = \overline{0} +_2 \overline{0} = f(a) +_2 f(b)$$

$$f(a.b) = \overline{0} = \overline{0} \cdot_2 \overline{0} = f(a) \cdot_2 f(b)$$

∴ f is homeomorphism.

Example: $f: Z \to Z_e \ni f(a) = 2a$

Solution:

$$f(a+b) = 2(a+b) = 2a + 2b$$

$$= f(a) + f(b)$$

LECTURE 8

$$\neq f(a.b) = 2(ab)$$

∴ f isn't homo.

Theorem: Let $f: R_1 \to R_2$ be a ring homo. then (1) $f(0) = 0_2 \ni 0_2$ identity of R_2

Proof:

$$a + 0 = 0 + a = a \ \forall a \in R$$

$$\therefore f(a+0) = f(0+a) = f(a)$$

$\therefore f(a) +_2 f(0) = f(0) +_2 f(a) = f(a)$ (f homo.)

$\therefore f(0)$ is identity in R_2.

But 0_2 is identity in R_2(unique)

$\therefore f(0) = 0_2$

Def: Let $f: F_1 \to F_2$ be a function. Then f is filed homo. if

1. $f(a+b) = f(a) +_2 f(b)$
2. $f(a.b) = f(a) ._2 f(b)$

Def: Let $f: R_1 \to R_2$ be a ring homo. then Kernel of f define by.

$$Ker(f) = \{a \in R : f(a) = 0_2\}$$

Remark:

1.
$Ker(f) \neq \emptyset$

Example: $f: R_1 \to R_2 \ni f(a) = 0_2, a \in R$.

$$Ker(f) = \{a \in R : f(a) = 0_2\}$$
$$= \{a : a \in R\} = R$$

Example: $f: R_1 \to R_2 \ni f(a) = a$

$$Ker(f) = \{a \in R : f(a) = 0\}$$
$$= \{ \qquad \}$$

Def: A field F is called prime filed if it has no

proper subfield. Example: Q, Z_3, Z_5

Remarks:

1. Any Quotient field of integral domain is a prime field
2. Any prime field is a quotient ideal.

Example: Let F be a field a field. Then 0 is maximal ideal and F is a minimal ideal

Example: $(Z, +, .)$

Z_e maximal

$(1) \supset (2)$

$(1) \supset I \supset (2)$

Theorem: Let $f: R_1 \rightarrow R_2$ be a ring homo. Then Ker(f) is an ideal in R.

Proof:

$Ker \neq \emptyset$

Let $x, y \in Ker(f)$

$\Longrightarrow f(x) = 0_2 , f(y) = 0_2$

T.P: $x - y \in Ker(f)$

T.P: $f(x - y) = 0_2$

$f(x - y) = f(x) - f(y) = 0_2 - 0_2 = 0$

Let $x \in Ker(f) , r \in R$

T.P: $rx , xr \in Ker(f)$

$f(rx) = f(r) . f(x) = f(r) . 0_2 = 0_2$

$\Longrightarrow rx \in Ker(f)$

$xr \in Ker(f)$

$\therefore Ker(f)$ is ideal in R.

Theorem: if $f: R_1 \rightarrow R_2$ is a ring home. Then f is one to one $\Leftrightarrow Ker(f) = 0$

Proof:

Suppose f is one to one

T.P $Kre(f) = \{0\}$

Let $a \in Ker(f)$

$\Longrightarrow f(a) = 0_2$

f

$(a) = f(0_2)$

$\therefore a = 0 \rightarrow Ker(f) = \{0\}$

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$\Leftarrow$ Suppose $Ker(f) = 0$

T.P: f is 1 -1

Let $a, b \in R_1 \ni f(a) = f(b)$

T.P: $a = b$

$f(a) = f(b)$

$f(a) - f(b) = 0$

$f(a - b) = 0 \to a - b \in Ker(f)$

$\Rightarrow a = b$

Corollary: The unique homo. otherwise, zero homo. is from $Z \to Z$ and it is identity ho
$(f(n) = n , n \in Z)$

Proof:

Let f be a non-zero

homo. from $Z \to Z$

T.P: f is identity

homo.

$(f(n) = n , n \in Z)$

Let $n \in {}^{+}Z$

$n = 1 + 1 + 1 + 1 + \cdots + 1$

$\therefore f(1) = f(1 + 1 + \ldots + 1)$

$= f(1) + f(1) + \cdots + f(1) = n.f(1)$

If n is negative $(n < 0)$

$\Longrightarrow -n \in Z^{+} \Longrightarrow f(n) = f\big(-(-n)\big) = -f(-n)$

$= -(-n)f(1) = n.f(1)$

If $n = 0 \to f(n) = f(0) = 0 = 0\, f(1)$

$\therefore f(n) = nf(1), n \in Z$

$\because f(1) = 1 \rightarrow f(n) = n \Longrightarrow f$ unique homo.

Def: Let $(I, +, .)$ Be an ideal of $(R, +, .)$. We define the set $\boldsymbol{ann(I)}$ by:

$$ann(I) = \{r \in R\,; r.a = 0, \forall a \in I\}$$

Q: prove that $(ann(I), +, .)$ Is an ideal of $(R, +, .)$

Proof:

Let:

$a = r_1 x \Longrightarrow a = 0$

$b = r_2 y \Longrightarrow b = 0$

1- $a - b \Longrightarrow r_1 x - r_2 y = 0 - 0 = 0$

2- $ra \Longrightarrow r0 = 0 \; . \forall r \in R$

Def: Let R be a ring. Then R is called Boolean ring if R has identity and $a^2 = a, \forall a \in R$. Ex: $(Z_2, +_2, ._2)$ is a Boolean ring because $Z_2 = \{0, 1\}$

$\Longrightarrow 0^2 = 0, 1^2 = 1$

Ex: $(P(X), \Delta, \cap)$ is a Boolean ring because $\forall A \in P(X) \Rightarrow A^2 = A \cap A = A$

Ex: $R = \{f : f : X \rightarrow Z_2\}$ and we define

$(f + g)(x) = f(x) +_2 g(x)$

$(fg) = f(x) ._2 g(x) \qquad \forall x \in X$

$\therefore$ R is a commutive ring with 1 and satisfy:

$$f(x) = 0 \;\; or \; f(1) = 1 \;\; , f \in R$$

$\therefore f^2 = f$

If $f(x) = 0 \Longrightarrow f^2(x) = f(x) ._2 f(x) = 0 ._2 0 = f(x)$

If $f(x) = 1 \Longrightarrow f^2(x) = f(x) \cdot_2 f(x) = 1 \cdot_2 1 = f(x)$

$$\therefore f^2 = f$$

∴ R is a Boolean ring

LECTURE 10

Theorem: Every Boolean ring is commutative and has Char = 2

Proof:

Suppose that R is a ring

and $a^2 = a \quad \forall\, a \in R$ Let a,

$b \in R \Longrightarrow a + b \in R$. why?

$\Rightarrow a^2 = a\ , b^2 = b\ , (a+b)^2 = a + b$

$\rightarrow (a+b)^2 = a^2 + b^2 + ab + ba$

$\rightarrow a + b = a^2 + b^2 + ab + ba$

$= a + b + ab +$
ba

$\rightarrow ab + ba = 0$

$\rightarrow a.a + a.a = 0$

a

${}^2 + a^2 = 0$

a

$+ a = 0$

$2a = 0 \ \ \forall a \in R$

$\therefore$ Char(R) = 2

Also, $ab + ba = 0 \Longrightarrow$

$ab + ab + ba = ab$

$2ab + ba = ab$

$0 + ba = ab$

$\Longrightarrow$

ab

$=$

ba

$\forall a,$

$b \in$

R $\therefore$

R

com

m.

Theorem: If R is a Boolean ring and I an ideal (proper) of R, then I is a prime ideal ⟺ I is maximal ideal.

Proof:

Suppose

that I is

prime ideal

T.P: I is

maximal??

Assume that I ideal

of R $\ni$ $\subset$ $\subseteq I\ J\ R$

T.P: $J = R$??

$\because$ $\subset I\ \ J \rightarrow \exists$ $\in$ $a\ J\ , a \notin I$

R Boolean

$\therefore a^2 = a$

$a(1-a) = 0 \in I$

$\because$ I prime $a \notin I$

$\therefore 1-a \in J$

$\because \subset I \; J \Longrightarrow 1-a \in J$

$\rightarrow (1-a)+a \in J \rightarrow 1 \in J \rightarrow J = R$

$\therefore$ I maximal

$\Leftarrow$

Suppose that I is maximal ideal

T.P: I is prime ideal

By "Every maximal ideal is prime ideal"

$\therefore$ I is prime ideal

LECTURE 11

Theorem: Let be a proper ideal of Boolean ring R. Then I is a maximal $\Leftrightarrow \frac{R}{I} \cong Z_2$

Proof:

Since R is a Boolean ring $\rightarrow \frac{R}{I}$ is a Boolean ring

Since R is a Boolean ring

$\rightarrow$ R is a commutative with 1

$\rightarrow \frac{R}{I}$ is a commutative with 1

T.P: $(0+I)^2 = a+I \quad , \forall\, a + I \in \frac{R}{I}$

I

$(a+I)^2 \in \frac{R}{I}$

$(a+I)^2 = (a+I)(a+I) = a.a + I = a^2 + I$

$= a + I \quad (a^2 = a)$

$\rightarrow \frac{R}{I}$ Boolean ring

The Boolean ring R is a filled $\Leftrightarrow R \cong Z_2$

Proof:

Let R be a Boolean ring.

$\therefore \; \forall \; \in \quad a \; R \rightarrow a = a.1$

$= a.(a.a^{-1})$

$= (a.a)a^{-1}$

$= a^2 . a^{-1}$

$= a.a^{-1} = 1$

$\because R = \{0, 1\}$

$\Longrightarrow R$

$\cong Z_2$

But I maximal $\Leftrightarrow \frac{R}{I}$ field

I maximal $\Leftrightarrow \frac{R}{I}$ Boolean ring

I maximal $\Leftrightarrow \frac{R}{I} \cong Z_2$

Theorem: Every Boolean ring R is semi simple

Proof:

Let R be a Boolean ring

$\therefore$ R has identity

element.

Why?? and a^2

$= a\ \forall a \in R$.

T. P. R is a semi simple ring

T. P.

$\therefore \exists$ homomorphism
from R to Z_2

$\ni f(a) = 1$

$\therefore Ker(f)$ ideal

(proper) in R

$\therefore \exists$ maximal

ideal M in R $\ni$

$Ker(f) \subseteq M$

But $1 - a \in Ker(f)$

$\therefore 1 - a \in M$ $(Ker(f) \subseteq M)$

Since $a \in M$ (because $a \in Rad(R)$)

$Rad\ (R) = \bigcap\{$ Maximal ideal$\}$

$\rightarrow 1 - a + a \in M$

$\rightarrow 1 \in M \rightarrow M = R$ C!

Because M Maximal in R

$\therefore Rad(R) = \{0\}$

R semi simple ring

Definition: We define Boolean algebra is a Mathematical system $(B, \vee, \wedge)$ such that $\vee, \wedge$ two binary operation on B and B$\neq \emptyset$ and satisfy the following

LECTURE 12

- $\vee, \wedge$ commutative on B.
  i.e.: $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ $\forall a, b \in B$.
- $\vee, \wedge$ distributive with them ; others ;
  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $\exists$ 1 , 0 identity elements with $\vee, \wedge$
  $\ni a \vee 0 = a$ & a
  $\wedge 1 = a$ $\forall a \in$
  B For each
  element a
  $\in$ B $\exists$ a' $\in$ B
  $\ni a \vee a' = 1$ & $a \wedge a' = 0$
  (a' is called complement of a)

Example: $(P(X), \cup, \cap)$, $x \neq 0$ is a Boolean algebra.
$0 = \emptyset$, $1 = x$

Example: Let B be a set of Positive integer numbers which

represent divisors of 10 $B = \{1,2,5,10\}$

We

define

$\vee, \wedge$

on B by:

$\forall$ a,b$\in$ B

$\rightarrow$

g.c.d (a, b) =$a \wedge b$

L.c.m (a, b) = a $\vee$ b . Then

(B , $\vee$, $\wedge$) is a Boolean algebra

$\vee$	1	2	5	10
1	1	2	5	10
2	2	2	10	10
5	5	10	5	10
10	10	10	10	10

$\wedge$	1	2	5	10
1	1	1	1	1
2	1	2	1	2
5	1	1	5	5
10	1	2	5	10

identity

element of

$\vee$ is 1

identity

element of

$\wedge$ is 10

1' = 10 , 2' = 5 , 5' = 2 , 10' = 1